Constructing Theta function solutions to the MEW equation based on symbolic computation

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Abstract

An accurate solution of the nonlinear partial differential equation has been the focus of studies for many years. This solution is helpful for understanding complex physics phenomena and dynamic processes. An auxiliary equation represented by theta functions is constructed by using the auxiliary equation method, which is applied to the modified equal width function. Double periodical wave solutions are numerically simulated using the accurate solution obtained by the symbolic computation software Mathematica. Results revealed that the auxiliary equation method is an effective and powerful mathematic tool for solving nonlinear evolution equations in mathematical physics using Mathematical.

Keywords: auxiliary equation method, MEW equation, theta function solutions

1 Introduction

The modified equal width (MEW) equation derived from nonlinear media through a dispersion process has been the focus of studies in the past decades [1–3]. Several methods have been presented to study the different solutions and physical phenomena related to this equation because of its wide applications and important mathematical properties. S. I. Zaki provided the solitary wave interactions of the MEW equation by using the Petrov–Galerkin method together with quintic B-spline finite elements. A. M. Wazwaz considered different solutions using the tanh and the sine–cosine methods, respectively. In addition, B. Saka provided numerical solutions by using collocation method. The MEW equation is presented as follows:

$$u_t + u^2 u_x - u_{xxt} = 0 . \tag{1}$$

Searching and constructing exact solutions for nonlinear partial differential equations (NLPDEs) is an ongoing research topic. These exact solutions can help in understanding the mechanism of complex physics phenomena and dynamic processes modelled by these NLPDEs. Several studies on exact solutions to NLPDEs exist, such as the famous inverse scattering method, the Back Lund transformation, the Darboux transformation, the HI rota bilinear method, and the Pain eve method [4–9]. Direct search for exact solutions to NLPDEs has become increasingly attractive in recent years partly because of the availability of symbolic computation systems such as Maple or Mathematica. These systems enable us to perform complicated and tedious algebraic calculations on a computer and help us find exact MEW solutions to NLPDE, such as the homogeneous balance method [10], the *tanh* function method [11], the sine–cosine method [12], the Jacobi elliptic functions method [13], the F-expansion method [14], and so on [15–19]. In this study, we apply the auxiliary equation method [20] to seek exact solutions to the MEW equation (Eq. 1) by taking full advantage of the elliptical equation

$$\left(F'(x)\right)^2 = b_4 F^4(x) + b_2 F^2(x) + b_0 . \tag{2}$$

and obtain traveling wave solutions in terms of theta functions with the aid of symbolic computation for the first time.

This paper is organized as follows: In the second section, we illustrate the auxiliary equation method and the properties of theta functions. In the third section, we apply the auxiliary equation method and an MEW solution of elliptical Eq. (2) to seek exact solutions of the MEW equation. The last section presents the conclusions.

2 Auxiliary equation method

We describe the auxiliary equation method as follows: Consider a given NLPDE with independent variables,

$$x = \left(x_1; x_2; \cdots; x_l; t\right)$$

and dependent variable u:

$$P\left(u; u_t; u_{x_1}; u_{tt}; \cdots\right) = 0. \tag{3}$$

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Generally speaking, the left-hand side of Eq. (3) is a polynomial in u and its various partial derivatives. We seek its traveling wave solution in the formal solution

$$u(\xi) = a_0 + \sum_{i=1}^{N} a_i F^n(\xi) \tag{4}$$

by taking

$$u(x_1; x_2; \cdots; x_l; t) = u(\xi),$$

$$\xi = k_1 x_1 + k_2 x_2 + \cdots + k_l x_l - \lambda t, \tag{5}$$

where

$k_1, k_2, \cdots k_l, \lambda$

and

$a_i (i = 1, 2, \cdots N)$

are constants to be determined, and

$F(\xi)$

satisfies elliptical Eq. (2).

Inserting (4) into (3) yields an ODE for

$u(\xi)$:

$$P(u; u'; u''; \cdots) = 0, \tag{6}$$

where integer N can be determined.

The following fact is needed to achieve the aim of this study for elliptical Eq. (2).

Proposition: If we take

$b_4 = -b_0 = -\vartheta_2^2(0)\vartheta_4^2(0)$

and

$b_2 = \vartheta_2^2(0) - \vartheta_4^2(0)$,

then

$F(z) = \vartheta_1(z)/\vartheta_3(z)$

satisfies elliptical Eq. (3), where theta functions are defined as follows:

$$\vartheta[\begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix}](z|\tau) = \sum_{n=-\infty}^{\infty} \exp\{\pi i \tau (n + \frac{\varepsilon}{2})^2 + 2(n + \frac{\varepsilon}{2})(z + \frac{\varepsilon'}{2})\}$$

$$\vartheta_i(z) \triangleq \vartheta_i(z|\tau) = \vartheta[\varepsilon_i](z|\tau), \tag{7}$$

$$\varepsilon_1 = \begin{bmatrix}1\\1\end{bmatrix}, \varepsilon_2 = \begin{bmatrix}1\\0\end{bmatrix}, \varepsilon_3 = \begin{bmatrix}0\\0\end{bmatrix}, \varepsilon_4 = \begin{bmatrix}0\\1\end{bmatrix},$$

where i=1,2,3,4.

The steps to determine $u(\xi)$ are as follows:

Step 1: N is determined by considering the homogeneous balance between the governing nonlinear term(s) and the highest-order derivatives of $u(\xi)$ in Eq. (6).

Step 2: Eq. (4) is substituted into Eq. (6), and Eq. (2) is used to convert the left-hand side of Eq. (6) into a finite series in:

$F^k(\xi)(k = 0, 1, \cdots, M)$.

Step 3: Equating each coefficient of $F^k(\xi)$ to zero produces a system of algebraic equations for $a_i (i = 0, 1, \cdots, N)$.

Step 4: With the aid of Mathematica or Maple in solving the system of algebraic equations, a_i, k_i, λ can be expressed by *A*, *B*, *C* (or the coefficients of ODE [Eq. 6]).

Step 5: Substituting these results into Eq. (5), we can obtain the general form of traveling wave solutions to Eq. (3).

Step 6: We can provide a series of theta function solutions to Eq. (3) from the proposition.

3 Exact Solutions to the MEW equation

In this section, we will use the auxiliary equation method and symbolic computation to find the exact solutions to the MEW equation.

We assume that Eq. (2) has a traveling wave solution in the form:

$$u(x,t) = U(\xi), \quad \xi = \rho x + \omega t. \tag{8}$$

Substituting Eq. (8) into Eq. (2), Eq. (2) is transformed into the following form:

$$\omega u' + \rho u^2 u' - \rho^2 \omega u''' = 0. \tag{9}$$

According to Step 1 in the second section, we obtain $n = 1$ by balancing u''' and uu' in Eq. (9). Suppose that Eq. (9) has the following solutions:

$$U(\xi) = a_0 + a_1 F(\xi), \tag{10}$$

$$U'(\xi) = a_1 F'(\xi), \quad U'''(\xi) = a_1 F''(\xi), \tag{11}$$

Substituting Eqs. (10) And (11) along with Eq. (3) into Eq. (9) produces a polynomial equation in $F(\xi)$. Setting their coefficients to zero obtains a set of algebraic equations with unknown parameters a_0, a_1, ω:

$$\omega a_1 b_0 + \rho a_0^2 a_1 b_0 - \rho^2 \omega a_1 b_0 b_2 = 0$$

$$\rho a_1^3 b_0 + \omega a_1 b_2 + \rho a_0^2 a_1 b_2 - \rho^2 \omega a_1 b_2^2$$

$$-6\rho^2 \omega a_1 b_0 b_4 = 0$$

$$\rho a_1^3 b_2 + \omega a_1 b_4 + \rho a_0^2 a_1 b_4 - 7\rho^2 \omega a_1 b_2 b_4 = 0$$

$$\rho a_0 a_1^2 b_0 = 0$$

$$\rho a_0 a_1^2 b_2 = 0$$

$$3\rho a_1 a_2 b_2 + \omega a_1 b_4 + \rho a_1 a_0 b_4 - 7\rho^2 \omega a_1 b_2 b_4 = 0$$

$$\rho a_1^3 b_4 - 6\rho^2 \omega a_1 b_4^2 = 0$$

By solving these equations with the use of the symbolic computation software Mathematica, we can obtain the following solutions:

$$a_1 = \frac{\sqrt{6\omega b_4}}{\sqrt[4]{b_2}}, a_0 = 0, \rho = \frac{1}{\sqrt{b_2}}, \tag{12}$$

$$a_1 = -\frac{i\sqrt{6\omega b_4}}{\sqrt[4]{b_2}}, a_0 = 0, \rho = -\frac{1}{\sqrt{b_2}}, \tag{13}$$

$$a_1 = \frac{i\sqrt{6\omega b_4}}{\sqrt[4]{b_2}}, a_0 = 0, \rho = -\frac{1}{\sqrt{b_2}}, \tag{14}$$

$$a_1 = -\frac{\sqrt{6\omega b_4}}{\sqrt[4]{b_2}}, a_0 = 0, \rho = \frac{1}{\sqrt{b_2}}, \qquad (15)$$

and substituting Eqs. (11)–(15) into Eq. (10), we obtain the traveling wave solutions of the MEW equation

$$u_1(x,t) = \frac{\sqrt{6\omega b_4}}{\sqrt[4]{b_2}} F^2(\xi),$$

where $\xi = \rho x + \omega t$, b_0, b_2, b_4, ω are arbitrary constants, and ρ is given in Eq. (12);

$$u_2(x,t) = -\frac{i\sqrt{6\omega b_4}}{\sqrt[4]{b_2}} F(\xi),$$

where $\xi = \rho x + \omega t$, b_0, b_2, b_4, ω are arbitrary constants, and ρ is given in Eq. (13);

$$u_3(x,t) = \frac{i\sqrt{6\omega b_4}}{\sqrt[4]{b_2}} F(\xi),$$

where $\xi = \rho x + \omega t$, b_0, b_2, b_4, ω are arbitrary constants, and ρ is given in Eq. (14);

$$u_4(x,t) = -\frac{\sqrt{6\omega b_4}}{\sqrt[4]{b_2}} F(\xi),$$

where $\xi = \rho x + \omega t$, b_0, b_2, b_4, ω are arbitrary constants, and ρ is given in Eq. (15).

If we choose $b_4 = -b_0 = -\vartheta_2^2(0)\vartheta_4^2(0)$ and $b_2 = \vartheta_2^2(0) - \vartheta_4^2(0)$ from the proposition, we can obtain solutions to the MEW equation in terms of theta functions

$$u_1(x,t) = \frac{\sqrt{6\omega b_4}}{\sqrt[4]{b_2}} \frac{\vartheta_1(\xi)}{\vartheta_3(\xi)},$$

where $\xi = \rho x + \omega t$, b_0, b_2, b_4, ω are arbitrary constants, and ρ is given in Eq. (12);

$$u_2(x,t) = -\frac{i\sqrt{6\omega b_4}}{\sqrt[4]{b_2}} \frac{\vartheta_1(\xi)}{\vartheta_3(\xi)},$$

where $\xi = \rho x + \omega t$, b_0, b_2, b_4, ω are arbitrary constants, and ρ is given in Eq. (13);

$$u_3(x,t) = \frac{i\sqrt{6\omega b_4}}{\sqrt[4]{b_2}} \frac{\vartheta_1(\xi)}{\vartheta_3(\xi)},$$

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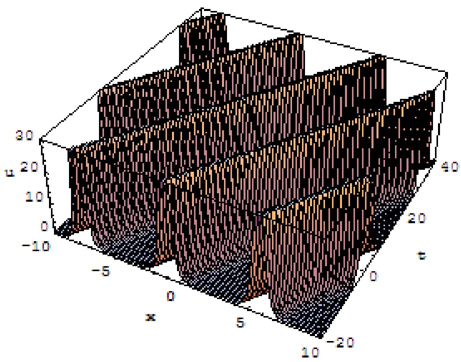
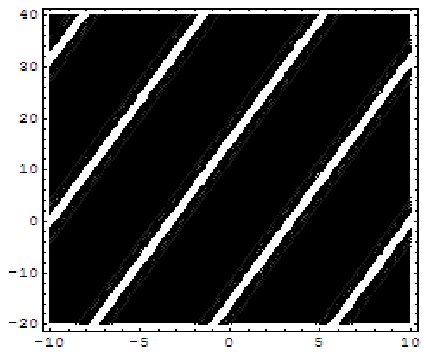
where $\xi = \rho x + \omega t$, b_0, b_2, b_4, ω are arbitrary constants, and ρ is given in Eq. (15).

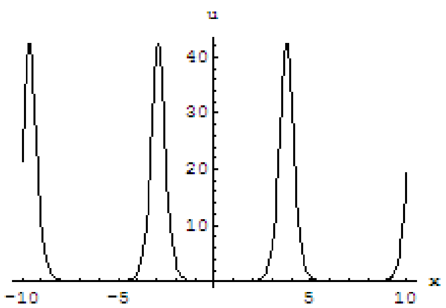
We depict the Figure 1 of the solution by using Mathematica to grasp the characteristics of Eq. (2) solutions.



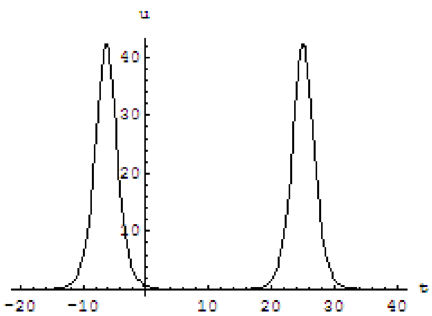
(a) Perspective view of the wave $u_1(x,t)$.



(b) Overhead view of the wave $u_1(x,t)$.



(c) Propagation of the wave along the x -axis.



(d) Propagation of the wave along the t -axis.

FIGURE 1 The solution $u_1(x,t)$: 1(a)–1(d)

Figures 1(a)–1(d) show the properties and profiles of Mathematica under the following parameters: $\omega = -0.1, \tau = 0.5$, and $t = 2$ for 2D Figure 1(c), and $x = 2$ for 2D Figure 1(d). Figures 1(a)–1(d) show that the solution $u_1(x,t)$ is a doubly periodic wave solution.

4 Conclusions

In this study, we examined the MEW equation. Some traveling wave solutions in terms of theta functions are successfully obtained by using the auxiliary equation method with the aid of symbolic computation for the first time. These solutions should be significant in explaining some physics phenomena. The auxiliary equation method is a very effective and powerful mathematical tool for solving nonlinear evolution equations in mathematics and physics. Moreover, it can be conveniently operated with the aid of symbolic computation systems such as Mathematica or Maple.

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